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Global solutions and asymptotic behaviors of the Chern–Simons–Dirac equations in $\mathbb{R}^{1+1}$

Hyungjin Huh

Department of Mathematics, Chung-Ang University, 221, Heukseok-dong, Dongjak-gu, Seoul, 156-756, Republic of Korea

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ABSTRACT

The initial value problem of the Chern–Simons–Dirac equations in one space dimension is studied. We prove the existence of global solution and investigate asymptotic behaviors.

1. Introduction

Since the introduction of the Chern–Simons action [9] as a new possible gauge field theory, it has been successfully applied to explain several phenomena such as the fractional quantum Hall effect and high temperature superconductivity. Moreover various field theories which include Chern–Simons terms are found to admit interesting classical soliton solutions. Recently it has been found that fermionic field theories coupled with the Chern–Simons gauge field admit vortex solutions [7,16].

The initial value problem of the Chern–Simons–Dirac equations in Minkowski space time $\mathbb{R}^{2+1}$ has been studied in [14]. The low regularity local in time solution was discussed by using Strichartz and null form estimates. In this paper we are interested in global solutions and their asymptotic properties of the Chern–Simons–Dirac system in one space dimension which is proposed as mathematical model equation.

We are working on the Minkowski space time with metric $g_{\mu\nu} = \mathrm{diag}(1, -1)$. From now on, the summation convention will be used for summing over repeated indices. Greek indices are used to denote 0, 1. We denote space time derivatives by $\partial_0 = \partial_t$, $\partial_1 = \partial_x$.

We consider the initial value problem of the Chern–Simons–Dirac equations in $\mathbb{R}^{1+1}$

$$i\gamma^\mu D_\mu \Psi = m\Psi, \tag{1.1}$$

$$\partial_t A_1 - \partial_x A_0 = \Psi^\dagger \alpha \Psi, \tag{1.2}$$

$$\partial_t A_0 - \partial_x A_1 = 0, \tag{1.3}$$

$$\Psi(0, x) = \psi(x), \qquad A_\mu(0, x) = a_\mu(x), \tag{1.4}$$

where Ψ denotes a 2-spinor field defined on $\mathbb{R}^{1+1}$. It is represented as a column vector with 2 components. $\Psi^\dagger = (\bar{\Psi}_1, \bar{\Psi}_1)$ denotes the complex conjugate transpose of Ψ. $A_\mu \in \mathbb{R}$ is the gauge field and $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative

E-mail address: huhmath@anmail.net.

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associated with the gauge field A_μ. The constant m is a mass parameter and α denotes (2×2) matrices which will be specified later. The Dirac gamma matrices γ^μ satisfy the anticommutation relation:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I,$$

where I is identity matrix and $\gamma^{0\dagger} = \gamma^0$, $\gamma^{1\dagger} = -\gamma^1$. One natural representation is a "Dirac" representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

System (1.1)–(1.3) has the conservation of charge

$$\|\Psi(t,\cdot)\|_{L^2(\mathbb{R})} = \|\Psi(0,\cdot)\|_{L^2(\mathbb{R})}. \tag{1.5}$$

To get an idea of the scaling property of the Chern–Simons–Dirac system, consider the massless case $m = 0$. Then Eqs. (1.1)–(1.3) are invariant under the rescaling

$$\Psi^\lambda(t,x) = \lambda \Psi(\lambda t, \lambda x), \qquad A^\lambda_\mu(t,x) = \lambda A_\mu(\lambda t, \lambda x).$$

Therefore the scale invariant data space is $\dot{H}^{-1/2}(\mathbb{R})$. Since the charge corresponds to the L^2 norm of the spinor, we may say that the initial value problem of (1.1)–(1.3) is charge subcritical.

The Dirac system coupled with Maxwell dynamics has been studied by several authors [2–6,8,10–13]. Especially, the Maxwell–Dirac system in one space dimension has been studied in [2–5,8]. The existence of global solutions to Maxwell–Dirac equations has been proved in [3,8] for the initial data $A(0,x) \in H^1(\mathbb{R})$, $\partial_t A(0,x) \in L^2(\mathbb{R})$ and $\Psi \in H^1(\mathbb{R})$. The several properties of the system have also been studied in [4]. Compared with Maxwell–Dirac system having quadratic nonlinear terms, Eqs. (1.2), (1.3) imply

$$\Box A_0 = \partial_x\big(\Psi^\dagger \alpha \Psi\big), \qquad \Box A_1 = \partial_t\big(\Psi^\dagger \alpha \Psi\big),$$

which have derivative quadratic terms making analysis difficult. Moreover Eqs. (1.1)–(1.3) show different features corresponding to algebraic properties of the quadratic terms $\Psi^\dagger \alpha \Psi$ which is important in our study. From now on, α denotes I or γ^0. The following are our main results.

Theorem 1.1 *(Case of $\alpha = I$). For the initial data $\psi \in H^1(\mathbb{R})$ and $a_\mu \in H^1(\mathbb{R})$, the initial value problem for* (1.1)–(1.4) *has a unique, global in time solution which belongs to*

$$\Psi \in C\big([0,\infty); H^1(\mathbb{R})\big), \qquad A_\mu \in C\big([0,\infty); H^1(\mathbb{R})\big).$$

The same idea as Theorem 1.1 can be applied to prove the existence of global solutions to (1.1)–(1.4) with $\alpha = \gamma^0$. Making use of the special algebraic structures of the nonlinear terms $A_\mu \gamma^\mu \Psi$ and $\Psi^\dagger \gamma^0 \Psi$, we can obtain a solution of low regularity.

Theorem 1.2 *(Case of $\alpha = \gamma^0$). For the initial data $\psi \in L^2(\mathbb{R})$ and $a_\mu \in L^2(\mathbb{R})$, there exists $T > 0$ such that the initial value problem for* (1.1)–(1.4) *has a unique, local in time solution*

$$\Psi \in C\big([0,T); L^2(\mathbb{R})\big), \qquad A_\mu \in C\big([0,T); L^2(\mathbb{R})\big).$$

Remark. It is interesting problem to prove global well-posedness of L^2 local solution which we call charge solution. Also local well-posedness issue below the charge norm is interesting.

For the mass zero case $m = 0$, we can show several asymptotic behaviors of the solutions by finding a representation formula for the solution to (1.1)–(1.4) in terms of the initial data. For instance, a decay of local charge can be shown.

Theorem 1.3. *For any $-\infty < l < r < \infty$, the smooth solutions Ψ, A_μ of* (1.1)–(1.4) *with $m = 0$ satisfies the decay of local L^2 norm*

$$\int_l^r \big|\Psi(t,x)\big|^2 \, dx \to 0, \quad \text{as } t \to \infty.$$

Moreover we have a pointwise decay for a fixed point x_0,

$$\lim_{t\to\infty} \big|\Psi(t,x_0)\big|^2 = 0.$$

Theorem 1.1 is proved in Section 2 and Theorem 1.2 in Section 3. We show in Section 4 Theorem 1.3 and other asymptotic behaviors of the solutions. We conclude this section by giving a few notations. We use the standard Sobolev spaces $W^{s,p}(\mathbb{R})$, $H^s(\mathbb{R}) = W^{s,2}(\mathbb{R})$ and $\dot{H}^s(\mathbb{R})$ with the norm $\|f\|_{\dot{H}^s} = \|(-\Delta)^{s/2} f\|_{L^2}$. Define the space time norm $\|f\|_{L_t^p L_x^q([0,T]\times\mathbb{R})} = (\int_0^T \|f(t,\cdot)\|_{L^q(\mathbb{R})}^p\,dt)^{1/p}$. The wave operator $\partial_{tt} - \partial_{xx}$ is denoted by $\Box$. The Schwartz class function is denoted by $\mathcal{S}(\mathbb{R}^n)$. We will use c, C to denote various constants. When we are interested in local solutions, we may assume that $T \leqslant 1$. Thus we shall replace smooth function of T, $C(T)$ by C. We use $A \lesssim B$ to denote an estimate of the form $A \leqslant CB$.

2. Proof of Theorem 1

In this section we restrict ourselves to the case $\alpha = I$. Let us recall some preliminaries to show the local well-posedness of the initial value problem of (1.1)–(1.4).

Lemma 2.1. *Let Ψ be the solution of*

$$i\gamma^\mu \partial_\mu \Psi = F, \quad \Psi(0,x) = g(x),$$

where $g \in H^s(\mathbb{R})$, $F \in L^1_{loc}(\mathbb{R}, H^s(\mathbb{R}))$. Then for any $T > 0$ and any $t \in [0,T]$ we have

$$\|\Psi(t,\cdot)\|_{H^s(\mathbb{R})} \leqslant c\left(\|g\|_{H^s(\mathbb{R})} + \int_0^T \|F(s,\cdot)\|_{H^s(\mathbb{R})}\,ds\right).$$

To estimate A_μ, we make use of the energy estimate. For the system

$$\begin{aligned} &\partial_t A_1 - \partial_x A_0 = G, \\ &\partial_t A_0 - \partial_x A_1 = 0, \end{aligned} \tag{2.1}$$

the following lemma is easily deduced.

Lemma 2.2. *For the smooth solution of* (2.1) *with sufficiently rapid decreasing condition at spatial infinity, the following energy inequality can be obtained*

$$\|A(t,\cdot)\|_{H^s(\mathbb{R})} \leqslant c\left(\|A(0,\cdot)\|_{H^s(\mathbb{R})} + \int_0^T \|G(s,\cdot)\|_{H^s(\mathbb{R})}\,ds\right).$$

Applying Lemmas 2.1 and 2.2 together with the fact $H^{\frac{1}{2}+\epsilon}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, the local well-posedness of (1.1)–(1.4) can be shown for the initial data ψ, a_μ belonging to $H^{\frac{1}{2}+\epsilon}(\mathbb{R})$. In fact the local existence result can be improved using the technique of Section 3. However, we are satisfied here with the solutions $\Psi, A_\mu \in C([0,T), H^{\frac{1}{2}+\epsilon}(\mathbb{R}))$ which are enough to go next issue.

From the above argument, local well-posedness of the solution to (1.1)–(1.4) can be guaranteed for the initial data ψ, $a_\mu \in H^1(\mathbb{R})$. To extend local solution globally we will bound $H^1(\mathbb{R})$ norm of Ψ and A_μ. Eq. (1.1) can be rewritten as

$$D_0\Psi + \gamma^0\gamma^1 D_1\Psi = -im\gamma^0\Psi. \tag{2.2}$$

Taking ∂_1 on both sides of (2.2), multiplying by $\partial_1\Psi^\dagger$ and getting a real part, we have

$$\frac{1}{2}\partial_t|\partial_1\Psi|^2 + \frac{1}{2}\partial_1\big(\partial_1\Psi^\dagger\gamma^0\gamma^1\partial_1\Psi\big) = \partial_1 A_0 \operatorname{Im}\big(\Psi^\dagger\partial_1\Psi\big) + \partial_1 A_1 \operatorname{Im}\big(\Psi^\dagger\gamma^0\gamma^1\partial_1\Psi\big). \tag{2.3}$$

Note that problematic terms $A_0\partial_1\Psi^\dagger\partial_1\Psi$ and $A_1\partial_1\Psi^\dagger\gamma^0\gamma^1\partial_1\Psi$ are cancelled out in (2.3) which is important to get global bound of H^1 norm. On the other hand, taking ∂_1 and multiplying (1.2), (1.3) by $\partial_1 A_1$, $\partial_1 A_0$ respectively, we have

$$\frac{1}{2}\partial_t|\partial_1 A|^2 - \partial_1 A_1\partial_1\partial_1 A_0 - \partial_1 A_0\partial_1\partial_1 A_1 = \partial_1 A_1\partial_1\big(\Psi^\dagger\Psi\big), \tag{2.4}$$

where $|\partial_1 A|^2 = (\partial_1 A_0)^2 + (\partial_1 A_1)^2$. Integrating (2.3) and (2.4) by parts on $\mathbb{R}$, we obtain

$$\begin{aligned} \frac{1}{2}\partial_t\int_{\mathbb{R}} |\partial_1\Psi|^2 + |\partial_1 A|^2 &= \int_{\mathbb{R}} \partial_1 A_0 \operatorname{Im}\big(\Psi^\dagger\partial_1\Psi\big) + \partial_1 A_1 \operatorname{Im}\big(\Psi^\dagger\gamma^0\gamma^1\partial_1\Psi\big) + \partial_1 A_1\,\partial_1\big(\Psi^\dagger\Psi\big) \\ &\lesssim \|\Psi\|_{L^\infty(\mathbb{R})}\big(\|\partial_1\Psi\|^2_{L^2(\mathbb{R})} + \|\partial_1 A\|^2_{L^2(\mathbb{R})}\big), \end{aligned} \tag{2.5}$$

where $\|\partial_1 A\|^2_{L^2(\mathbb{R})} = \|\partial_1 A_0\|^2_{L^2(\mathbb{R})} + \|\partial_1 A_1\|^2_{L^2(\mathbb{R})}$.

To apply Gronwall's lemma to (2.5), we bound $\|\Psi\|_{L^\infty}$ as follows. Eq. (1.1) can be rewritten as

$$\partial_t\Psi_1 + \partial_1\Psi_2 = iA_0\Psi_1 + iA_1\Psi_2 - im\Psi_1,$$
$$\partial_t\Psi_2 + \partial_1\Psi_1 = iA_0\Psi_2 + iA_1\Psi_1 + im\Psi_2.$$

Putting $g_1 = \Psi_1 + \Psi_2$ and $g_2 = \Psi_1 - \Psi_2$, we have

$$\partial_t g_1 + \partial_1 g_1 = i(A_0 + A_1)g_1 - img_2, \tag{2.6}$$
$$\partial_t g_2 - \partial_1 g_2 = i(A_0 - A_1)\,g_2 - img_1. \tag{2.7}$$

Multiplying (2.6) by $\bar{g}_1$ and taking real part, we obtain

$$\partial_t|g_1|^2 + \partial_x|g_1|^2 = 2m\,\mathrm{Im}(\bar{g}_1 g_2). \tag{2.8}$$

By applying similar process to (2.7), we have

$$\partial_t|g_2|^2 - \partial_x|g_2|^2 = 2m\,\mathrm{Im}(\bar{g}_2 g_1). \tag{2.9}$$

From (2.8) and (2.9) we derive the integral equations

$$|g_1|^2(t,x) = |g_1|^2(0,x-t) + 2m\int_0^t \mathrm{Im}(\bar{g}_1 g_2)(s, x-t+s)\,ds,$$
$$|g_2|^2(t,x) = |g_2|^2(0,x+t) + 2m\int_0^t \mathrm{Im}(\bar{g}_2 g_1)(s, x+t-s)\,ds. \tag{2.10}$$

Taking now $\|\cdot\|_{L^\infty(\mathbb{R})}$ on Eqs. (4.5) and applying Gronwall's lemma, we have for some positive constant c depending on the initial data and m,

$$\big\||g_1|^2(t,\cdot)\big\|_{L^\infty(\mathbb{R})} + \big\||g_2|^2(t,\cdot)\big\|_{L^\infty(\mathbb{R})} \lesssim e^{ct}, \tag{2.11}$$

from which we derive $\|\Psi(t,\cdot)\|_{L^\infty(\mathbb{R})} \lesssim e^{ct}$. Now applying Gronwall's lemma to (2.5), we conclude that

$$\big\|\partial_1\Psi(t,\cdot)\big\|^2_{L^2(\mathbb{R})} + \big\|\partial_1 A(t,\cdot)\big\|^2_{L^2(\mathbb{R})} \lesssim \big(\|\partial_1\psi\|^2_{L^2(\mathbb{R})} + \|\partial_1 a\|^2_{L^2(\mathbb{R})}\big)\exp\big(ce^{ct}\big),$$

which shows the upper bound of $\dot{H}^1$ norm of Ψ, A_μ. Considering (1.5) we conclude that H^1 norms of $\Psi(t,\cdot)$, $A_\mu(t,\cdot)$ are bounded for $t \in [0,T]$. Therefore H^1 local solution can be extended globally past given time interval $[0,T]$.

3. Proof of Theorem 2

In this section we are interested in the case of $\alpha = \gamma^0$. The similar idea to the proof of Theorem 1.1 can be applied to show the global existence of H^1 solution of (1.1)–(1.4) with $\alpha = \gamma^0$. An interesting problem is the existence of low regularity solution. The solutions of the Dirac–Klein–Gordon system with low regularity has been investigated in [1,2,17,18]. Their proofs are based on the observation that the quadratic nonlinear terms are related to the null forms. To get low regularity solution of (1.1)–(1.4), the special algebraic properties of the quadratic terms $A_\mu\gamma^\mu\Psi$ and $\Psi^\dagger\gamma^\mu\Psi$ will be made use of which are related with the following null conditions:

$$Q_0(u,v) = \partial_t u\partial_t v - \partial_x u\partial_x v,$$
$$Q_1(u,v) = \partial_t u\partial_x v - \partial_x u\partial_t v,$$

for functions $u(t,x)$ and $v(t,x)$. The novel point of null structures is that if the solutions Ψ and A_μ belong to $C([0,T], L^2(\mathbb{R}))$, then the quadratic nonlinearities $A_\mu\gamma^\mu\Psi$ and $\Psi^\dagger\gamma^0\Psi$ become $L^2([0,T]\times\mathbb{R})$. To see that we introduce the following lemma in [2].

Lemma 3.1. *Let u, v be the solutions to the following initial value problems,*

$$\Box u = F, \qquad u(0,x) = f_0, \qquad \partial_t u(0,x) = f_1,$$
$$\Box v = G, \qquad v(0,x) = g_0, \qquad \partial_t v(0,x) = g_1,$$

and let Q be any of the null forms. Then

$$\|Q(u,v)\|_{L^2([0,T]\times\mathbb{R})} \leqslant c\left(\|f_0\|_{H^1(\mathbb{R})} + \|f_1\|_{L^2(\mathbb{R})} + \int_0^T \|F(s,\cdot)\|_{L^2(\mathbb{R})}\,ds\right) \times \left(\|g_0\|_{H^1(\mathbb{R})} + \|g_1\|_{L^2(\mathbb{R})} + \int_0^T \|G(s,\cdot)\|_{L^2(\mathbb{R})}\,ds\right).$$

To observe the null structures, we introduce, for given functions Ψ and A_μ, the solutions of the following equations

$$\begin{aligned} &\gamma^\mu \partial_\mu \Phi = \Psi, \\ &\partial_0 B_1 - \partial_1 B_0 = A_0, \\ &\partial_0 B_0 - \partial_1 B_1 = -A_1, \\ &\Phi(0,x) = 0, \qquad B_\mu(0,x) = 0. \end{aligned} \tag{3.1}$$

Making use of (3.1) we can check

$$\Psi^\dagger \gamma^0 \Psi = Q_0(\bar{\Phi}_1, \Phi_1) - Q_0(\bar{\Phi}_2, \Phi_2) + Q_1(\bar{\Phi}_1, \Phi_2) - Q_1(\bar{\Phi}_2, \Phi_1), \tag{3.2}$$

and

$$A_0\gamma^0\Psi + A_1\gamma^1\Psi = \begin{pmatrix} Q_0(B_1,\Phi_1) + Q_0(B_0,\Phi_2) + Q_1(B_0,\Phi_1) + Q_1(B_1,\Phi_2) \\ Q_0(B_0,\Phi_1) + Q_0(B_1,\Phi_2) + Q_1(B_0,\Phi_2) + Q_1(B_1,\Phi_1) \end{pmatrix}. \tag{3.3}$$

Moreover we have

$$\begin{aligned} &\Box\Phi = \gamma^\mu \partial_\mu \Psi = i A_\mu \gamma^\mu \Psi - im\Psi, \\ &\Box B_0 = -\partial_0 A_1 + \partial_1 A_0 = -\Psi^\dagger \gamma^0 \Psi, \\ &\Box B_1 = 0. \end{aligned} \tag{3.4}$$

We are ready to prove Theorem 1.2 which follows by standard arguments from a priori estimates of the following propositions.

Proposition 3.2. *Let (Ψ, A_μ) be a solution of Eqs.* (1.1)–(1.4) *with $\alpha = \gamma^0$ in a strip $[0,T]\times\mathbb{R}$ with*

$$A_\mu \in C\big([0,T]; L^2(\mathbb{R})\big), \qquad \Psi \in C\big([0,T]; L^2(\mathbb{R})\big).$$

Define

$$J(T) = \sup_{0\leqslant t\leqslant T} \|\Psi(t,\cdot)\|_{L^2(\mathbb{R})} + \|A_\mu(t,\cdot)\|_{L^2(\mathbb{R})}.$$

Then there exist constants $C > 0$ and $T^* > 0$, depending only on $J(0)$, such that if $T < T^*$ then $J(T) \leqslant CJ(0)$.

Proposition 3.3. *Let (Ψ, A_μ) and (Ψ', A'_μ) be two solutions of Eqs.* (1.1)–(1.4) *with $\alpha = \gamma^0$ verifying the hypothesis of Proposition* 3.2 *in a strip $[0,T]\times\mathbb{R}$ and let $J(T)$ as in Proposition* 3.2 *and $J'(T)$ be the corresponding quantity for the primed solution. Define*

$$\triangle(T) = \sup_{0\leqslant t\leqslant T} \big(\|\Psi(t,\cdot) - \Psi'(t,\cdot)\|_{L^2(\mathbb{R})} + \|A_\mu(t,\cdot) - A'_\mu(t,\cdot)\|_{L^2(\mathbb{R})}\big).$$

Then there exist constants $C > 0$ and $T^* > 0$, depending only on $J(0)$ and $J'(0)$, such that if $T < T^*$ then $\triangle(T) \leqslant C\triangle(0)$.

Proposition 3.3 follows by the similar argument to Proposition 3.2. We will only present the proof of Proposition 3.2. We define

$$X(T) = \int_0^T \|A_\mu\gamma^\mu\Psi(s,\cdot)\|_{L^2(\mathbb{R})} + \|\Psi\gamma^0\Psi(s,\cdot)\|_{L^2(\mathbb{R})}\,ds.$$

It is easily shown that $J(T) \lesssim J(0) + X(T)$ by applying Lemmas 2.1 and 2.2. We will derive the inequality $X(T) \lesssim T^{1/2}(J(0) + X(T))^2$. Then a bootstrap argument completes the proof of Proposition 3.2.

The first integral of $X(T)$ can be treated considering (3.3), (3.4) and Lemma 3.1.

$$\begin{aligned}
\int_0^T \|A_\mu \gamma^\mu \Psi(s,\cdot)\|_{L^2}\,ds &\leqslant T^{1/2}\|Q(B,\Phi)\|_{L^2([0,T]\times\mathbb{R})} \\
&\lesssim T^{1/2}\left(\|\partial_t B(0,\cdot)\|_{L^2} + \int_0^T \|\Box B\|_{L^2}\,ds\right)\left(\|\partial_t \Phi(0,\cdot)\|_{L^2} + \int_0^T \|\Box \Phi\|_{L^2}\,ds\right) \\
&\lesssim T^{1/2}\left(J(0) + \int_0^T \|\Psi\gamma^0\Psi\|_{L^2}\,ds\right)\left(J(0) + \int_0^T \|A_\mu\gamma^\mu\Psi\|_{L^2}\,ds\right) \\
&\lesssim T^{1/2}\big(J(0)+X(T)\big)^2.
\end{aligned}$$

The second quantity of $X(T)$ is also bounded as follows.

$$\begin{aligned}
\int_0^T \|\Psi^\dagger \gamma^0 \Psi(s,\cdot)\|_{L^2}\,ds &\leqslant T^{1/2}\|Q(\Phi,\Phi)\|_{L^2([0,T]\times\mathbb{R})} \\
&\lesssim T^{1/2}\left(J(0) + \int_0^T \|A_\mu\gamma^\mu\Psi\|_{L^2}\,ds\right)\left(J(0) + \int_0^T \|A_\mu\gamma^\mu\Psi\|_{L^2}\,ds\right) \\
&\lesssim T^{1/2}\big(J(0)+X(T)\big)^2.
\end{aligned}$$

Therefore we get the relation $X(T) \lesssim T^{1/2}(J(0)+X(T))^2$ which completes the proof of Proposition 3.2 by a bootstrap argument.

4. Asymptotic behaviors

Here we prove several properties of the global solution to the Chern–Simons–Dirac equations (1.1)–(1.4) by deriving an explicit representation formula. The similar idea was applied to the Maxwell–Dirac system [15]. From now on we assume massless case $m = 0$ and the solutions we consider belong to $\bigcap_{n=0}^{\infty} H^n((0,T)\times\mathbb{R})$ for a given T. By Sobolev's embedding theorem they become $C^\infty((0,T)\times\mathbb{R})$ solutions which satisfy equations in the classical sense.

First we prove a decay of local charge of the solution. From Eqs. (2.6), (2.7) with $m = 0$, we have

$$\partial_t g_1 + \partial_x g_1 = i(A_0 + A_1)g_1, \tag{4.1}$$

$$\partial_t g_2 - \partial_x g_2 = i(A_0 - A_1)g_2. \tag{4.2}$$

Integrating (4.1), (4.2) along outgoing and ingoing characteristic respectively, we obtain

$$g_1(t,x) = g_1(0,x-t)\exp\left(i\int_0^t (A_0+A_1)(s,x-t+s)\,ds\right),$$

$$g_2(t,x) = g_2(0,x+t)\exp\left(i\int_0^t (A_0-A_1)(s,x+t-s)\,ds\right),$$

which are equivalent to

$$(\Psi_1+\Psi_2)(t,x) = (\psi_1+\psi_2)(x-t)\exp\left(i\int_0^t (A_0+A_1)(s,x-t+s)\,ds\right),$$

$$(\Psi_1-\Psi_2)(t,x) = (\psi_1-\psi_2)(x+t)\exp\left(i\int_0^t (A_0-A_1)(s,x+t-s)\,ds\right). \tag{4.3}$$

Then we deduce

$$\big|(\Psi_1+\Psi_2)(t,x)\big| = \big|(\psi_1+\psi_2)(x-t)\big|,$$
$$\big|(\Psi_1-\Psi_2)(t,x)\big| = \big|(\psi_1-\psi_2)(x+t)\big|,$$

which implies

$$\left|\Psi_1(t,x)\right|^2 + \left|\Psi_2(t,x)\right|^2 = \frac{1}{2}\left(\left|(\psi_1+\psi_2)(x-t)\right|^2 + \left|(\psi_1-\psi_2)(x+t)\right|^2\right). \tag{4.4}$$

Integrating (4.4) on finite interval (l, r) we can check a decay of the local charge in spite of the conservation of total charge (1.5)

$$\int_l^r \left(|\Psi_1|^2 + |\Psi_2|^2\right)(t,x)\,dx \to 0 \quad \text{as } t \to \infty.$$

For a fixed point x_0, we can check a pointwise decay

$$\lim_{t\to\infty} \left(|\Psi_1|^2 + |\Psi_2|^2\right)(t,x_0) = 0,$$

which proves Theorem 1.3.

In the rest of this section we complete solution representation for Eqs. (1.1)–(1.4). For the linear equations

$$\partial_t A_1 - \partial_x A_0 = F,$$
$$\partial_t A_0 - \partial_x A_1 = 0,$$
$$A_0(0,x) = a_0(x), \qquad A_1(0,x) = a_1(x),$$

we have a solution representation

$$(A_0 + A_1)(t,x) = a_0(x+t) + a_1(x+t) + \int_0^t F(s, -s+t+x)\,ds,$$
$$(A_0 - A_1)(t,x) = a_0(x-t) - a_1(x-t) - \int_0^t F(s, s+x-t)\,ds. \tag{4.5}$$

Now we assume $\alpha = I$ in (1.2). Recalling $\Psi^\dagger\Psi(t,x) = |\Psi_1(t,x)|^2 + |\Psi_2(t,x)|^2$ and considering (4.4), (4.5) we obtain

$$(A_0 + A_1)(t,x) = a_0(x+t) + a_1(x+t) + \frac{t}{2}\left|(\psi_1-\psi_2)(t+x)\right|^2 + \frac{1}{2}\int_0^t \left|(\psi_1+\psi_2)(t+x-2s)\right|^2 ds,$$
$$(A_0 - A_1)(t,x) = a_0(x-t) - a_1(x-t) - \frac{t}{2}\left|(\psi_1+\psi_2)(x-t)\right|^2 - \frac{1}{2}\int_0^t \left|(\psi_1-\psi_2)(x-t+2s)\right|^2 ds. \tag{4.6}$$

Then Eqs. (4.3) and (4.6) give a desired solution representation for (1.1)–(1.4) with $m = 0$ and $\alpha = I$.

Making use of it, we are able to observe several properties of the solution to the Chern–Simons–Dirac equations. For the initial data $\psi(x)$, $a_\mu(x) \in \mathcal{S}(\mathbb{R})$, Eqs. (4.6) give

$$\lim_{t\to\infty} 2A_1(t,x) = \lim_{t\to\infty} \frac{1}{4}\int_{x-t}^{x+t} \left|(\psi_1+\psi_2)(s)\right|^2 + \left|(\psi_1-\psi_2)(s)\right|^2 ds = \frac{1}{2}\int_{-\infty}^{\infty} \left|\psi_1(s)\right|^2 + \left|\psi_2(s)\right|^2 ds.$$

Then we have shown

Corollary 4.1. *If Ψ, A_μ is a classical solution of* (1.1)–(1.4) *($m = 0$ and $\alpha = I$) for a nontrivial initial data $\psi \in \mathcal{S}(\mathbb{R})$, then we have for each $x \in \mathbb{R}$*

$$\lim_{t\to\infty} A_1(t,x) = \frac{1}{4}\|\psi\|^2_{L^2(\mathbb{R})}.$$

This result can be used to show that Eqs. (1.1)–(1.4) do not have a scattering theory.

Corollary 4.2. *There is no free solution with initial data in $\mathcal{S}(\mathbb{R})$ to which the global solution of* (1.1)–(1.4) *with $m = 0$, $\alpha = I$ tends in $H^1(\mathbb{R})$.*

Proof. Suppose (Ψ^+, A_μ^+) is the free solution which is asymptotically similar to the given solution. Then Sobolev's embedding theorem says

$$\left\| A_1(t,\cdot) - A_1^+(t,\cdot) \right\|_{L^\infty(\mathbb{R})} \lesssim \left\| A_1(t,\cdot) - A_1^+(t,\cdot) \right\|_{H^1(\mathbb{R})} \to 0 \quad \text{as } t \to \infty.$$

However we know $\lim_{t\to\infty} A_1^+(t,x) = 0$ for a given x because it has data in $\mathcal{S}(\mathbb{R})$ while we have $\lim_{t\to\infty} A_1(t,x) \geqslant c > 0$ by Corollary 4.2, which gives the contradiction. $\square$

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